

**A – SETS.**

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According to E. Borel [2] a set  $X$  of real numbers has *strong measure zero* if there is for each sequence  $\epsilon_1, \epsilon_2, \dots, \epsilon_n, \dots$  of positive real numbers a corresponding sequence  $I_1, I_2, \dots, I_n, \dots$  of open intervals such that  $\text{length}(I_n) < \epsilon_n$  for each  $n$ , and  $X \subseteq \bigcup_{n=1}^{\infty} I_n$ . The collection of strong measure zero sets forms a  $\sigma$ -ideal. Galvin, Mycielski and Solovay [4] discovered that the notion of a strong measure zero set has an algebraic characterization:  $X$  has strong measure zero if, and only if, there is for each first category set  $M$  a real number  $t$  such that  $X \cap (t + M) = \emptyset$ .

By analogy a set  $Y$  was defined to have strong first category if there is for each measure zero set  $N$  a real number  $t$  such that  $Y \cap (t + N) = \emptyset$ . Some of the basic combinatorial properties of the collection of strong first category sets are not easily discernible from this algebraic definition: Only recently Pawlikowski proved that Sierpinski sets have strong first category. At present it is not even known if this collection of sets is an ideal!

In this connection, the first author and H. Judah, pursuing ideas of I. Reclaw, defined and studied properties of the so-called  $R^{\mathcal{M}}-$ ,  $SR^{\mathcal{M}}-$ ,  $R^{\mathcal{N}}-$  and  $SR^{\mathcal{N}}-$  sets [1]. For convenience of the reader we briefly define these notions here.  $\mathcal{M}$  denotes the ideal of first category subsets of the real line, while the symbol  $\mathcal{N}$  denotes the ideal of measure zero sets. Suppose that  $\mathcal{J}$  is an ideal of subsets of the real line. A Borel set  $H \subseteq \mathbb{R} \times \mathbb{R}$  is called  $\mathcal{J}$ -set if  $(H)_x = \{y : (x, y) \in H\} \in \mathcal{J}$  for all  $x \in \mathbb{R}$ . We say that  $X \subseteq \mathbb{R}$  is an  $R^{\mathcal{J}}$  set if for every  $\mathcal{J}$ -set  $H$ ,  $\bigcup_{x \in X} (H)_x \neq \mathbb{R}$ . A set  $X \subseteq \mathbb{R}$  is an  $SR^{\mathcal{J}}$  set if for every  $\mathcal{J}$ -set  $H$ ,  $\bigcup_{x \in X} (H)_x \in \mathcal{J}$ .

Independently, the second author started searching for covering properties of the collection of strong first category sets, analogous to the properties studied by Menger [6], Hurewicz [5] and Rothberger [7].

A common thread seems to be emerging from these two approaches. In this paper we report some of our findings to illustrate this. We introduce what we call  $A$  – sets. The motivation for the name is simply that letters from the alphabet seem to be used for various small sets studied by various authors (such as  $C$ -sets,  $Q$ -sets, and so on) and the letter  $A$  had not been reserved yet.

For a set  $X$  in the range of a function  $f$ , we write  $f^{\leftarrow}[X]$  to denote the pre-image of  $X$  under  $f$ . Where not specified, our notation or terminology follows that of [1] or [3].

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1991 *Mathematics Subject Classification.* 03E20, 04A99.

*Key words and phrases.* Menger-, Hurewicz- and Rothberger- properties.

The second author was supported by Idaho State Board of Education grant 94-051.

1. THE  $A_1$ -SETS.

Let  $X$  be a set of real numbers.  $X$  is an  $A_1$ -set if: For every sequence  $(\mathfrak{U}_1, \mathfrak{U}_2, \dots)$  such that  $X \subseteq \bigcup \mathfrak{U}_n$  for each  $n$ , and each  $\mathfrak{U}_n$  is a *countable* collection of Borel sets, there is a sequence  $(Y_1, Y_2, \dots)$  such that for each  $n$  we have  $Y_n \in \mathfrak{U}_n$  and  $X \subseteq \bigcup_{n < \infty} Y_n$ .

The  $A_1$ -property is a direct analogue of Rothberger's property  $C''$ . Indeed, we have:

**Theorem 1.** *For a set  $X$ , the following are equivalent:*

1.  $X$  is an  $A_1$ -set.
2. Every Borel image of  $X$  into  ${}^{\mathbb{N}}\mathbb{N}$  has Rothberger's property  $C''$ .
3.  $X$  is an  $R^{\mathcal{M}}$ -set.

*Proof.* (1)  $\Rightarrow$  (2) : Let  $\Psi : X \rightarrow {}^{\mathbb{N}}\mathbb{N}$  be a Borel function. Let  $(\mathfrak{U}_1, \mathfrak{U}_2, \dots, \mathfrak{U}_n, \dots)$  be a sequence of open covers of  $\Psi[X]$ ; we may assume that each is countable. Enumerate each  $\mathfrak{U}_n$  bijectively as  $\{U_1^n, U_2^n, \dots, U_k^n, \dots\}$ .

Then  $\Psi^{-1}[U_k^n] = F_k^n$  is a Borel subset of  $X$  for each  $(n, k)$ . Choose Borel subsets  $A_k^n$  of  $\mathbb{R}$  such that  $F_k^n = X \cap A_k^n$ . For each  $n$  put  $\mathcal{S}_n = \{A_k^n : k \in \mathbb{N}\}$ . Then each  $\mathcal{S}_n$  is a cover of  $X$  by countably many Borel sets. Select for each  $n$  a  $k_n$  such that  $X \subseteq \bigcup_{n=1}^{\infty} A_{k_n}^n$ . Then  $\Psi[X] \subseteq \bigcup_{n=1}^{\infty} U_{k_n}^n$ .

(2)  $\Rightarrow$  (1) : Assume that every Borel image of  $X$  into  ${}^{\mathbb{N}}\mathbb{N}$  has property  $C''$ . Let  $(\mathfrak{U}_1, \mathfrak{U}_2, \dots, \mathfrak{U}_n, \dots)$  be a sequence such that each  $\mathfrak{U}_n$  is a covering of  $X$  by countably many Borel sets. Enumerate  $\mathfrak{U}_n$  bijectively as  $\{U_1^n, U_2^n, \dots, U_k^n, \dots\}$ . Define

$$\Psi : X \rightarrow {}^{\mathbb{N}}\mathbb{N}$$

so that for each  $x \in X$  we have  $\Psi(x)(n) = \min\{m : x \in U_m^n\}$ . Note that for every basic open set  $[(n_1, \dots, n_k)]$ ,  $\Psi^{-1}([(n_1, \dots, n_k)]) = X \cap (U_{n_1}^1 \cap \dots \cap U_{n_k}^k) \setminus (U_1^1 \cup \dots \cup U_{n_1-1}^1 \cup \dots \cup U_1^k \cup \dots \cup U_{n_k-1}^k)$ , a Borel subset of  $X$ . Then  $\Psi$  is a Borel mapping. By Hypothesis  $\Psi[X]$  has property  $C''$ . Thus, we find a  $g$  such that for each  $x$  the set  $\{n : \Psi(x)(n) = g(n)\}$  is infinite (see [9], Lemma 1). But then  $X \subseteq \bigcup_{n=1}^{\infty} U_{g(n)}^n$ .

As to the equivalence of (2) and (3): According to Theorem 4.3 of [1],  $X$  is an  $R^{\mathcal{M}}$ -set if, and only if, every Borel image of it into  ${}^{\mathbb{N}}\mathbb{N}$  has the Rothberger property.  $\square$

In particular, we see that every set which has property  $A_1$  also has Rothberger's property  $C''$ . Since for example the Sierpinski sets do not have measure zero, while  $C''$ -sets do, we see that the collection of  $A_1$ -sets is too small to capture all the sets of strong first category.

**Corollary 2.** *The collection of  $A_1$ -sets is a  $\sigma$ -ideal.*

*Proof.* It is clear from the definition of these sets that this collection is closed under countable unions. Being an  $R^{\mathcal{M}}$ -set is hereditary, as is clear from the definition of that notion.  $\square$

Since we have the inclusions  $SR^{\mathcal{N}} \subseteq SR^{\mathcal{M}} \subseteq R^{\mathcal{M}}$ , it follows that  $SR^{\mathcal{N}}$ -sets and  $SR^{\mathcal{M}}$ -sets are  $A_1$ -sets.

2. THE  $A_2$ -SETS.

In his 1927 paper [5] Hurewicz introduced a property of sets which is nowadays referred to as property  $H$ ; see [3]. A set  $X$  of real numbers has property  $H$  if for

every sequence  $\langle \mathfrak{G}_n : n < \omega \rangle$  of open covers of  $X$ , there exists a sequence  $\langle \mathfrak{F}_n : n < \omega \rangle$  such that each  $\mathfrak{F}_n$  is a finite subset of  $\mathfrak{G}_n$ , and such that  $X \subseteq \bigcup_{n=1}^{\infty} (\bigcap_{m=n}^{\infty} \mathfrak{F}_m)$ .

Let  $X$  be a set of real numbers. We say that  $X$  is an  $A_2$ -set if: For every sequence  $(\mathfrak{U}_1, \mathfrak{U}_2, \dots)$  such that  $X \subseteq \bigcup \mathfrak{U}_n$  for each  $n$ , and each  $\mathfrak{U}_n$  is a countable collection of Borel sets, there is a sequence  $(\mathfrak{G}_1, \mathfrak{G}_2, \dots)$  such that for each  $n$  we have  $\mathfrak{G}_n$  is a finite subset of  $\mathfrak{U}_n$  and  $X \subseteq \bigcup_{n=1}^{\infty} (\bigcap_{m=n}^{\infty} \mathfrak{G}_m)$ .

According to [1], a set has property  $\mathcal{H}$  if every Borel image of it in  ${}^{\mathbb{N}}\mathbb{N}$  is bounded. As noted in [1], every Sierpinski set is an  $R^{\mathcal{N}}$  set with property  $\mathcal{H}$ . It follows that  $R^{\mathcal{N}} \cap \mathcal{H} \not\subset A_1$ .

**Theorem 3.** *For a set  $X$ , the following are equivalent:*

1.  $X$  is an  $A_2$  - set.
2. Every Borel image of  $X$  into  ${}^{\mathbb{N}}\mathbb{N}$  is bounded.
3. Every Borel image of  $X$  into  ${}^{\mathbb{N}}\mathbb{N}$  has property  $H$ .

*Proof.* That (1) implies (2): Let  $\Psi : X \rightarrow {}^{\mathbb{N}}\mathbb{N}$  be a Borel function. Let  $\mathcal{O}_n = \{[\sigma] : \sigma \in {}^n\mathbb{N}\}$  for each  $n$ . Then  $(\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_n, \dots)$  is a sequence of open covers of  $\Psi[X]$ . Enumerate each  $\mathcal{O}_n$  bijectively as  $\{[\sigma_1^n], [\sigma_2^n], \dots, [\sigma_k^n], \dots\}$ . Then  $\Psi^{\leftarrow}[[\sigma_k^n]] \stackrel{\text{def}}{=} F_k^n$ , say) is a Borel subset of  $X$  for each  $(n, k)$ . For each  $(n, k)$  choose a Borel subset  $B_k^n$  of  $\mathbb{R}$  such that  $F_k^n = X \cap B_k^n$ .

For each  $n$ , put  $\mathfrak{U}_n = \{B_k^n : k \in \mathbb{N}\}$ . Then  $(\mathfrak{U}_1, \mathfrak{U}_2, \dots, \mathfrak{U}_n, \dots)$  is a sequence of countable Borel covers of  $X$ . Since  $X$  is an  $A_2$ -set, we find for each  $n$  a finite set  $\mathcal{V}_n \subset \mathfrak{U}_n$  such that  $X \subseteq \bigcup_{n=1}^{\infty} (\bigcap_{n=m}^{\infty} \mathcal{V}_n)$ . We may assume that there is for each  $n$  a  $k_n$  such that  $\mathcal{V}_n = \{B_j^n : j \leq k_n\}$ .

Putting  $g(n) = \max\{\sigma_j^{n+1}(n) : j \leq k_{n+1}\}$ , we see that  $\Psi(x) \prec g$  for each  $x \in X$ .

To see that (2) implies (3), consider any Borel function  $\Psi$  from  $X$  to  ${}^{\mathbb{N}}\mathbb{N}$ . For any continuous function  $h$  from  $\Psi[X]$  to  ${}^{\mathbb{N}}\mathbb{N}$ , the function  $h \circ \Psi$  is a Borel function on  $X$ , so that  $h \circ \Psi[X]$  is bounded. But this shows that every continuous image of  $\Psi[X]$  is bounded; by §5 of [5],  $\Psi[X]$  has property  $H$ .

To see that (3) implies (1), let  $(\mathfrak{U}_1, \mathfrak{U}_2, \dots, \mathfrak{U}_n, \dots)$  be a sequence such that for each  $n$ ,  $\mathfrak{U}_n$  is a countable cover of  $X$  with Borel sets. Enumerate each  $\mathfrak{U}_n$  bijectively as  $\{U_1^n, U_2^n, \dots, U_k^n, \dots\}$ . For each  $x \in X$ , define

$$\Psi(x)(m) = \min\{n : x \in U_n^m\}.$$

Then  $\Psi$  is a Borel mapping, and so  $\Psi[X]$  has property  $H$ . Since the identity mapping is continuous,  $\Psi[X]$  is a bounded subset of  ${}^{\mathbb{N}}\mathbb{N}$ . Choose  $g$  such that  $\Psi(x) \prec g$  for each  $x$ . Put  $\mathcal{V}_n = \{U_i^n : i \leq g(n)\}$  for each  $n$ . Then  $X \subseteq \bigcup_{n=1}^{\infty} (\bigcap_{m=n}^{\infty} \mathcal{V}_m)$ .  $\square$

We see that  $R^{\mathcal{N}} \cap \mathcal{H} \subset A_2$ ; in particular, every Sierpinski set is an  $A_2$ -set.

### 3. THE $A_3$ -SETS.

In his 1924 paper [6], Menger introduced a property of sets which is related to the Rothberger property. This Menger property is nowadays referred to as property  $M$  (see [3]). A set  $X$  of real numbers has property  $M$  if for every sequence  $\langle \mathfrak{G}_n : n < \omega \rangle$  such that each  $\mathfrak{G}_n$  is a chain of open sets covering  $X$ , there exists a sequence  $\langle F_n : n < \omega \rangle$  such that each  $F_n \in \mathfrak{G}_n$ , and such that  $\{F_n : n < \omega\}$  is an open cover of  $X$ .

The proof of Theorem 3 on p. 21 of [3] shows, *mutatis mutandis* (see also [9], Proposition 3), that

**Theorem 4.** *For a subset  $X$  of the real line, the following are equivalent:*

1.  $X$  has property  $M$ .
2. No continuous image of  $X$  into  ${}^\omega\omega$  is a dominating family.

We now introduce our third covering property. Let  $X$  be a first category set of real numbers.  $X$  is an  $A_3$ -set if: For every sequence  $(\mathfrak{U}_1, \mathfrak{U}_2, \dots)$  such that  $X \subseteq \cup \mathfrak{U}_n$  for each  $n$ , and each  $\mathfrak{U}_n$  is a countable chain of Borel sets, there is a sequence  $(Y_1, Y_2, \dots)$  such that for each  $n$  we have  $Y_n \in \mathfrak{U}_n$  and  $X \subseteq \cup_{n < \infty} Y_n$ . We denote the collection of  $A_3$ -sets by the symbol  $\mathcal{A}_3$ .

The collection  $\mathcal{A}_3$  is closed under countable unions.

**Theorem 5.** *For  $X$  a set of real numbers, consider the following assertions: (1), (2), (3) and (4) are equivalent and each implies (5).*

1.  $X$  has property  $A_3$ .
2. For every sequence  $(\mathfrak{U}_1, \mathfrak{U}_2, \dots, \mathfrak{U}_n, \dots)$  such that each  $\mathfrak{U}_n$  is a collection of countably many Borel sets whose union contains  $X$ , there is a sequence  $(\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_n, \dots)$  such that
  - $\mathcal{V}_n$  is a finite subset of  $\mathfrak{U}_n$  for each  $n$ , and
  - $X \subseteq \cup_{n=1}^\infty (\cup \mathcal{V}_n)$ .
3. Every Borel image of  $X$  in  ${}^\omega\omega$  has property  $M$ .
4. No Borel image of  $X$  in  ${}^\omega\omega$  is dominating.
5.  $X$  has property  $M$ .

*Proof.* The proof that (1) and (2) are equivalent is standard. To see that (1) implies (3), let  $X$  be a set with property  $A_3$ , and let  $\Psi$  be a Borel function from  $X$  to  ${}^\omega\omega$ . We must show that  $\Psi[X]$  has property  $M$ .

Take a sequence  $(\mathfrak{G}_n : n < \omega)$  of open covers of  $\Psi[X]$ . We may assume that each  $\mathfrak{G}_n$  is a countable ascending chain of open sets. Enumerate  $\mathfrak{G}_n$  in ascending order as  $\{G_1^n, G_2^n, \dots, G_m^n, \dots\}$ .

For each  $(n, m)$ , put  $F_m^n = \Psi^{-1}[G_m^n]$ , a Borel subset of  $X$ . Observe that  $F_m^n \subseteq F_{m+1}^n$  for all  $n$  and  $m$ . For each  $(n, m)$  we fix a Borel subset  $A_m^n$  of the real line such that  $F_m^n = X \cap A_m^n$ . We may assume that  $A_m^n \subseteq A_{m+1}^n$  for all  $n$  and  $m$ .

Define:  $\mathfrak{U}_n = \{A_1^n, A_2^n, \dots, A_k^n, \dots\}$ . Then each  $\mathfrak{U}_n$  is an ascending chain of Borel sets, and  $X \subseteq \cup \mathfrak{U}_n$  for each  $n$ .

Since  $X$  is a  $A_3$ -set, we find for each  $n$  a  $k_n$  such that  $X \subseteq \cup_{n=1}^\infty A_{k_n}^n$ . But then  $\Psi[X] \subseteq \cup_{n=1}^\infty G_{k_n}^n$ , and we succeeded in finding the required selector for the given sequence of open covers.

To see that (3) implies (1), let  $(\mathfrak{U}_n : n < \omega)$  be a sequence of ascending chains of Borel sets, such that for each  $n$  we have  $X \subseteq \cup \mathfrak{U}_n$ . For each  $n$  enumerate  $\mathfrak{U}_n$  in ascending order as  $\{U_1^n, U_2^n, \dots, U_k^n, \dots\}$ . Now define

$$\Psi : X \rightarrow {}^\mathbb{N}\mathbb{N}$$

so that  $\Psi(x)(n) = \min\{m : x \in U_m^n\}$ . Since for a finite sequence  $(i_1, \dots, i_n)$  of natural numbers we have

$$X \cap (U_{i_1}^1 \cap \dots \cap U_{i_n}^n) = \{x \in X : j \leq n \Rightarrow \Psi(x)(j) = i_j\},$$

we see that the inverse image of any basic open subset of  ${}^\mathbb{N}\mathbb{N}$  is a Borel subset of  $X$ . Thus  $\Psi$  is a Borel mapping.

Then  $\Psi[X]$  has property  $M$ . As subset of  ${}^{\mathbb{N}}\mathbb{N}$ , it is not a dominating family. Choose a  $g$  which is not dominated by the family  $\Psi[X]$ . For each  $x \in X$  we see that there is an  $n$  such that  $\Psi(x)(n) \leq g(n)$ . Finally we set  $V_n = U_{g(n)}^n$  for each  $n$ . Then  $V_n \in \mathfrak{U}_n$ , and  $X \subseteq \bigcup_{n=1}^{\infty} V_n$ .

To see that (3) implies (4), note that if  $\Psi$  is a Borel mapping and  $g$  is continuous, then  $g \circ \Psi$  is a Borel mapping, and the identity mapping  $I(x) = x$  of  ${}^{\omega}\omega$  is continuous. Thus, if a Borel image of  $X$  is dominating, we would have a subset of  ${}^{\omega}\omega$  which has property  $M$  and yet is dominating, a contradiction.

That (4) implies (3) can be seen as follows: consider a Borel image of  $X$ . It is not dominating. Following this Borel function with a continuous function results in a Borel function, and so this second image is still not dominating. Thus, no continuous image of the image of a Borel mapping of  $X$  into  ${}^{\omega}\omega$  is dominating.

To see that (4) implies (5), we simply note that continuous functions are Borel functions; thus no continuous image of  $X$  is dominating. But this implies that  $X$  has property  $M$ .  $\square$

**Corollary 6.** *Every  $A_1$ -set and every  $A_2$ -set is an  $A_3$  set.*

**Corollary 7.** *Property  $A_3$  is hereditary,  $\mathcal{A}_3$  is a  $\sigma$ -ideal.*

*Proof.* The property is hereditary on account of (4) of Theorem 5. Since  ${}^{\omega}\omega$  is a Borel image of  $\mathbb{R}$ , it also follows from Theorem 5 that  $\mathcal{A}_3$  is a  $\sigma$ -ideal.  $\square$

If  $X \subset Z$  is a subset of a topological space  $Z$ , and if  $D \subset Z$  is countable, then any continuous function  $\pi : X \rightarrow Y$  can be extended to an  $F_{\sigma}$  (and *ipso facto* a Borel) function  $\rho : X \cup D \rightarrow Y$ .

In [8] Reclaw shows that  $MA$  implies the existence of a set of real numbers  $X$  with the properties (see his Theorem 3):

- There exists a continuous function from  $X$  onto the closed unit interval,
- $X + F$  has Lebesgue measure zero whenever  $F$  does, and
- there is a countable set  $D$  such that  $X \cup D$  is a  $\gamma$ -set.

This set  $X$  cannot be an  $A_3$ -set. For the set  $Y = [0, 1] \setminus \mathbb{Q}$  is homeomorphic with  ${}^{\omega}\omega$  (via continued fraction expansion), and this homeomorphism can be extended to all of  $[0, 1]$  in such a way that the resulting map is  $F_{\sigma}$ . Then we have an  $F_{\sigma}$  function from  $X$  onto  ${}^{\omega}\omega$ . Indeed, we find an  $F_{\sigma}$  map from  $X \cup D$  onto  ${}^{\omega}\omega$ . Accordingly, null-additive (and thus first category-additive) sets need not be  $A_3$ -sets. In particular, strong first category sets or even first category sets of strong measure zero need not be  $A_3$  sets. Also,  $\gamma$ -sets need not be  $A_3$ -sets.

**Proposition 8.**  $\text{add}(\mathcal{A}_3) = \mathfrak{b}$ .

*Proof.* It is clear that  $\text{add}(\mathcal{A}_3)$  is regular and uncountable. Let  $\lambda < \mathfrak{b}$  be a regular infinite cardinal number, and assume that for each  $\mu < \lambda$  it had already been established that  $\mu < \text{add}(\mathcal{A}_3)$ . Let  $X_{\alpha}$ ,  $\alpha < \lambda$  be a sequence of elements of  $\mathcal{A}_3$  such that  $X_{\alpha} \subset X_{\beta}$  whenever  $\alpha < \beta$ . Put  $X = \bigcup_{\alpha < \lambda} X_{\alpha}$ . Let  $(\mathfrak{U}_1, \mathfrak{U}_2, \mathfrak{U}_3, \dots, \mathfrak{U}_n, \dots)$  be a sequence such that each  $\mathfrak{U}_n$  is an ascending chain of Borel sets whose union contains  $X$ . For each  $n$  we write  $\mathfrak{U}_n = \{U_1^n, U_2^n, \dots, U_m^n, \dots\}$  where  $U_m^n \subset U_k^n$  whenever  $m < k$ .

For each  $\alpha < \lambda$ , define a function  $f_{\alpha}$  such that  $X_{\alpha} \subseteq \bigcup_{n=1}^{\infty} U_{f_{\alpha}(n)}^n$ . Since we have  $\lambda < \mathfrak{b}$ , select a function  $f$  such that  $f_{\alpha} \prec f$  for each  $\alpha < \lambda$ .

Next, fix  $I \in [\lambda]^{\lambda}$  and  $N < \infty$  such that

1.  $m > N \Rightarrow f_\alpha(m) < f(m)$  for each  $\alpha \in I$ , and
2.  $f_\alpha \restriction_{N+1} = f_\beta \restriction_{N+1} = \sigma$  for all  $\alpha, \beta \in I$ .

Put  $M_1 = U_{\sigma(1)}^1, \dots, M_N = U_{\sigma(N)}^N$ , and  $M_k = U_{f(k)}^k$  for each  $k > N$ . Then  $X_\alpha \subseteq \cup_{k=1}^\infty M_k$  for each  $\alpha \in I$ , whence  $X \subseteq \cup_{k=1}^\infty M_k$ . This shows that  $\mathfrak{b} \leq \text{add}(\mathcal{A}_3)$ . To see that  $\text{add}(\mathcal{A}_3) \leq \mathfrak{b}$  is easy, and left to the reader.  $\square$

Thus,  $MA$  implies that every set of reals of cardinality less than  $\mathfrak{c}$  is a  $A_3$ -set. There are first category sets which do not have property  $A_3$ .

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